RESEARCH PROJECT: OBSERVABILITY OF WAVES AND SCHRÖDINGER EQUATIONS ON ROUGH DOMAINS

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This research project concerns the observability of two dispersive equations, strongly connected with actual physical models, namely the wave and Schrödinger equations. Observability means an estimation of the energy of a free solution by means of some localized measure (in space and time). The derivation of such observability estimates is often studied because of the following applications:

(1) Exact controllability, that is, driving the solution to a prescribed state in a prescribed time.
(2) Stabilization, that is, having the energy of the solution decay to zero by means of a damping term as time goes to infinity. Decay rate depends on the quality of the observability estimate.
(3) Inverse problems, that is, identifying parameters like coefficients by means of some measures on the solution.

This question has been studied on bounded domains in the case of smooth coefficients with methods that rely on microlocal analysis. With such methods, necessary and sufficient geometrical conditions can be obtained. They involve rays of geometrical optics. Microlocal methods are however usually quite greedy in terms of coefficient regularity or boundary regularity. For rougher boundaries, older methods apply. In fact, until the end of the 80's, most of the results were proven under a (global) geometrical assumption called Γ-condition and introduced by J.-L. Lions [11], essentially based on a multiplier method. Yet, such method only apply to the case of constant coefficients (or small perturbation of that case), which is very restrictive.

The main goal of the present project is to extend microlocal methods to the cases of rougher coefficients and boundaries. Recent results have given methods to achieve such a goal.

1. Review of the observability of wave and Schrödinger equations

In this presentation we first focus on the wave equation. Suppose $\mathcal{M}$ is a bounded open set of $\mathbb{R}^d$ or a manifold with boundary. Given a metric g, the Laplace-Beltrami opetator reads $\Delta_g = \operatorname{div}_g \nabla_g$, that is, in local coordinates

$$\Delta_g f = (\det g)^{-1/2} \sum_{1\leq i,j\leq d} \partial_{x_i}\big((\det g)^{1/2} g^{ij}(x)\partial_{x_j} f\big), \tag{1}$$

and the wave operator reads $P = \partial_t^2 - \Delta_g$. The wave equation reads

$$\begin{cases} Pu = 0 & \text{in } \mathbb{R}\times\mathcal{M},\\ u = 0 & \text{in } \mathbb{R}\times\partial\mathcal{M},\\ u_{|t=0} = \underline{u}^0,\ \partial_t u_{|t=0} = \underline{u}^1 & \text{in } \mathcal{M}, \end{cases} \tag{2}$$

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here in the form of a Cauchy problem with the initial conditions $\underline{u}^0$ and $\underline{u}^1$. A unique solution exists for instance if $\underline{u}^0 \in H_0^1(\mathcal{M})$ and $\underline{u}^1 \in L^2(\mathcal{M})$. Its energy is given by

$$\mathcal{E}(u)(t) = \frac{1}{2}\big(\|\nabla_g u(t)\|^2_{L^2(\mathcal{M})} + \|\partial_t u(t)\|^2_{L^2(\mathcal{M})}\big),$$

which can be proven constant in time t.

Given on open set Γ of the boundary $\partial\mathcal{M}$ and some $T > 0$, one says that **boundary observability** holds if there exists $C_{\text{obs}} > 0$ such that for any $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ one has

$$\mathcal{E}(u) \leq C_{\text{obs}} \big\| \mathbf{1}_{(0,T)\times\Gamma}\, \partial_{\mathsf{n}} u_{|\mathbb{R}\times\partial\mathcal{M}} \big\|^2_{L^2(\mathbb{R}\times\partial\mathcal{M})}, \tag{3}$$

for the solution u to (2).

Following Rauch and Taylor [13], Bardos, Lebeau and Rauch proved observability inequalities from part of the boundary in their seminal article [1], and as a consequence, boundary stabilization, under a microlocal condition, that is, a property in the cotangent bundle $T^*(\mathbb{R} \times \Omega)$, the so-called geometric control condition (GCC in short), exhibiting a connection between the set Γ on which observation is performed and the generalized geodesics of the wave operator. In addition, taking into account the work of [6], it is now classical that observability (with stability with respect to the observation set) is equivalent to the GCC. In terms of geodesics, the GCC reads as follows:

for any point x and any tangent vector v, the generalized geodesic initiated at (x, v) enters the observation region in some time $T > 0$.

Generalized geodesics follow the laws of geometrical optics at boundary points: reflection if the boundary is hit transversally and possible gliding if hit tangentially.

2. Regularity issues

The proofs of the results in [1, 6] are based on microlocal tools, namely, the propagation of wave-front sets or that of microlocal defect measures. Despite their high efficiency and robustness, these methods present the great disadvantage of requiring a lot of regularity for the domain and for the metric/coefficients. Starting from the original result developed in the framework of the Melrose-Sjöstrand $\mathscr{C}^\infty$-singularity propagation results, thus requiring $\mathscr{C}^\infty$-smoothness, the theory has been subsequently developed in the framework of microlocal defect measures [9, 10] allowing one to relax the assumptions down to a $\mathscr{C}^2$-metric [2], which barely misses the natural smoothness ($W^{2,\infty}$) required to define a geodesic flow (away from any boundary). Below this smoothness threshold, for instance for $\mathscr{C}^1$-metrics, generalized geodesics may still exist as integral curves of the $\mathscr{C}^0$-Hamiltonian vector field in the interior of the domain but *uniqueness is lost* in general. A natural question lies in the understanding of the relationship between those nonunique integral curves and the observability property for such a rough metric. This problem is treated in the recent work [4, 5]. A sufficient condition for observability to hold in the case of a $\mathscr{C}^1$-metric and $\mathscr{C}^2$-boundary is proven. It reads

for any point x and any tangent vector v, **any** *generalized geodesic initiated at (x, v) enters the observation region in some time $T > 0$.*

Note the slight difference with the former condition that takes into account non uniqueness.

The proof relies on a contradiction argument. One analyses concentration phenomena that can occur if observability does not hold. A semiclassical measure emerges from this analysis and one proves that this measure fulfills a transport equation along the Hamiltonian vector field associated with the wave operator. As mentionned above this vector field is only continuous yielding non trivial phenomena. One proves in [4, 5] that the support of the measure is a union of maximal generalized geodesics which allows one to obtain a contradiction with the new GCC condition.

3. Goals of the present project research

3.1. **Rougher boundaries.** The measure transport equation originates from [10]. Yet, no propagation result for the measure suppport was deduces from this equation in [10]. In [10] the setting is different than in [4, 5].

- in [10], the flat metric was used on a bounded open set Ω of $\mathbb{R}^d$ with a $W^{2,\infty}$-boundary.
- in [4, 5], the metric is $\mathscr{C}^1$ and the boundary is $\mathscr{C}^2$.

Even though $\mathscr{C}^2$ may seem close to $W^{2,\infty}$ a lot can happen if lowering the boundary regularity down to $W^{2,\infty}$. In particular, one cannot use change of coordinates. Indeed, in the new coordinates one cannot guarantee the metric to be more than Lipschitz, a disaster for the mere existence of geodesics.

A first goal will thus to investigate if propagation of support measure holds in the case of $\mathscr{C}^1$-metric on a bounded domain with $W^{2,\infty}$-boundary, meaning the extention of [4, 5] to an rougher boundary.

For a $W^{2,\infty}$ boundary, a second goal, will then be to obtain observability estimates in the case of a metric that is only Lipschitz yet close to a $\mathscr{C}^1$-metric. Such perturbative results have been proven in [3] in the case of a manifold without boundary and generalized in [4, 5] in the case of a $\mathscr{C}^2$-boundary.

3.2. **Other boundary conditions.** In (2) one uses homogeneous Dirichlet boundary conditions. One goal of the project is the investigation of more general Lopatinskii conditions in the spirit of [7]. This is an important topic as Dirichlet condition often simply the analysis.

Neumann boundary condition, that is, condition concerning the normal derivative of the solution at the boundary $\partial_{\mathsf{n}} u_{|\partial\mathcal{M}}$ are not encompassed by the general framework of Lopatinskii conditions in the case of the wave equation. A goal is also to study the above questions under this conditions. Results are known in that case only for smooth coefficients; see for instance [1].

3.3. **Schrödinger equation.** Above we focussed on the wave equation. Similar questions can be raised for the Schrödinger equation. One can also consider the semiclassical Schrödinger equation associated with the operator $h^2\Delta_g+V$, where h is the semiclassical parameter, to be associated with the Plank constant physically, and V is a potential. In the recent article [8], concentration phenomena for this operator are considered in the case of a singular yet structured potential V: conormal singularities are considered ($IW^{1,1}$ and $IW^{2,1}$) with respect to a smooth hypersurface Σ. The authors of [8] prove that the semiclassical measure is transported along the bicharacteristic flow yet under some restrictions:

(1) contacts of order 1 with Σ (hyperbolic case) are considered for $V \in IW^{1,1}$
(2) contacts of order 1 and 2 are allowed for $V \in IW^{2,1}$.

The project is to use the techniques developed in [4, 5] to extend those propagation results allowing for higher-order contacts (glancing case) or even allowing for geodesics to remain in the hypersurface for some time (gliding case).

3.4. **Case of semilinear equations.** Last but not least, extension of the results obtained in the case of linear equations to the case of semi-linear equation in the spirit of [12] is a goal that fits very well within this research program.

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