Proposition d'encadrement de thèse.

Thématique : **Algorithmique & Combinatoire**

Laboratoire : Laboratoire d'Informatique de Paris-Nord (LIPN, UMR CNRS 7030).
99, avenue Jean-Baptiste Clément 91430 Villetaneuse. France.

Équipe : *Combinatoire, algorithmique et interactions* (CALIN).

Nom et adresse électronique des directeurs de travaux :

V. Hoang Ngoc Minh, E-mail : Minh@lipn.univ-paris13.fr (directeur).

G.H.E. Duchamp, E-mail : gerard.duchamp@lipn.univ-paris13.fr (co-encadrant)

Nom et adresse électronique du directeur du laboratoire :

Frédérique Bassino. E-mail : frederique.bassino@lipn.univ-paris13.fr

Titre : *Towards Picard-Vessiot theory for noncommutative differential equations*

Picard-Vessiot (PV for short) theory of bilinear systems[1] was realized in [14]. This theory allows to exploit, with success, linear algebraic groups in control theory (*i.e.* as symmetry groups of linear differential equations), for which some questions were solved thanks to the theory of Hopf algebras [5] and some combinatorial and effective aspects were set in [22].

Let us, for instance, consider the following nonlinear dynamical system

$$\left\{\begin{array}{lcl} y(z) & = & f(q(z)), \\ \dot{q}(z) & = & \sum_{i\geq 1} A_i(q)\, u_i(z), \\ q(z_0) & = & q_0, \end{array}\right. \tag{1}$$

where the vector state $q=(q_1,\ldots,q_n)$ belongs to a complex analytic manifold $\mathcal{M}$ of dimension n, q_0 is the initial state, the observation function f belongs to $\mathcal{H}(U)$ where U is a fixed connected neighbourhood of[2] q_0, the vector fields $\{A_i\}_{i\geq 1}$ being defined as follows

$$A_i = \sum_{j=1}^{n} A_i^j(q)\frac{\partial}{\partial q_j}, \quad \text{with} \quad A_i^j(q) \in \mathcal{H}(U). \tag{2}$$

From [13], it is also known that the output of the nonlinear system (1) is obtained (under some convergence criterion [7, 8]) by:

$$y(z_0,z) = \sum_{i_1,\ldots,i_k=0}^{k} (A_{i_1}\circ\ldots\circ A_{i_k}[f])_{|q_0} \int_{z_0}^{z} u_{i_1}(z_1)dz_1 \ldots \int_{z_0}^{z_{k-1}} u_{i_k}(z_k)dz_k \tag{3}$$

i.e. as a pairing between,

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[1] Namely - locally - linear of the states $q_1,\ldots,q_N$ and linear of the inputs $u_0,\ldots,u_m$ [14].

[2] In this description the points are loosely identified with their coordinates through some chart $\varphi_U : U \to \mathbb{C}^n$ likewise the space of analytic functions $\mathcal{H}(U)$ is described by $\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$.

1. on the one hand, the generating series[3], noted $\sigma f_{|_{q_0}}$ [13], of the system (1) on the noncommutative variables over the alphabet $X = \{x_i\}_{i\geq 1}$ (the free monoid, generated by X, is denoted by X^* and is equipped 1_{X^*} as neutral element) :

$$\sigma f_{|_{q_0}} := \sum_{w\in X^*} (\mathcal{A}(w)[f])_{|_{q_0}} w \in \mathcal{H}(U)\langle\langle X\rangle\rangle, \tag{4}$$

   where $\mathcal{A}$ is morphism, for the concatenation, defined recursively by

$$\mathcal{A}(1_{X^*}) = \mathrm{Id} \text{ and } \mathcal{A}(vx_i) = \mathcal{A}(v)\circ A_i, \text{ for } v\in X^*, x_i\in X, \tag{5}$$

2. on the other hand, the Chen series, $C_{z_0\rightsquigarrow z}$, of the differential forms

$$\forall i\geq 1,\quad \omega_i(z) := u_i(z)dz, \tag{6}$$

   along a path $z_0 \rightsquigarrow z$ over a simply connected manifold[4] Ω:

$$C_{z_0\rightsquigarrow z} := \sum_{w\in X^*} \alpha_{z_0}^z(w)w \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle, \tag{7}$$

   where $\mathcal{H}(\Omega)$ denotes the ring of analytic functions over Ω (with neutral element $1_{\mathcal{H}(\Omega)}$) and, for dynamical subdivisions $(z_0, z_1, \ldots, z_k, z)$ of the path $z_0 \rightsquigarrow z$ in Ω, we define

$$\alpha_{z_0}^z(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \text{ and } \alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) = \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k). \tag{8}$$

In order to follow this route, we consider the differential ring $(\mathcal{H}(\Omega), \partial)$ and equip $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ with the differential operator defined by

$$\forall S\in \mathbb{C}.1_{\mathcal{H}(\Omega)}\langle\langle X\rangle\rangle, \qquad \mathbf{d}S = \sum_{w\in X^*} (\partial\langle S \mid w\rangle dz)w. \tag{9}$$

Hence, on the one side, the differential ring $(\mathcal{H}(\Omega)\langle\langle X\rangle\rangle, \mathbf{d})$ is such that $\mathrm{Const}(\mathcal{H}(\Omega)\langle\langle X\rangle\rangle) = \ker\mathbf{d} = \mathbb{C}.1_{\mathcal{H}(\Omega)}\langle\langle X\rangle\rangle$ and, on the other side, a PV theory of nonlinear systems (1) should be intimately connected with the following first order noncommutative differential equation

$$\mathbf{d}S = MS \quad \text{with} \quad M = \sum_{i\geq 1}\omega_i x_i. \tag{10}$$

which is considered as the universal differential equation, *i.e.* universality can be seen by replacing the letters x_i by matrices[5] (resp. analytic vector fields) and obtaining linear (resp. nonlinear) differential equations. It follows that the differential Galois group of (10) (acting on group-like solutions) is the Hausdorff group of $\mathcal{H}_{\sqcup\!\sqcup}(X)$. This leads us to define the PV extension related to (10) as $\widehat{\mathcal{C}_0.X}\{C_{z_0\rightsquigarrow z}\}$.

In this project, we propose to extend the previous results and to illustrate this theory, under construction, in the resolution of the following differential equation involved in the study of models in quantum field theory [11]

$$dF = \Omega_n F, \quad \text{for} \quad n\geq 2 \tag{11}$$

[3]This series is not necessarily rational [1, 18].
[4]Usually one dimensional manifold. It will be the support of the iterated integrals below.
[5]These matrices are typically variable but as the alphabet is not limited in size, in many cases, singularities can be sorted and matrices taken constant.

over $\mathbb{C}_*^n=\{(z_1,\ldots,z_n)\in\mathbb{C}^n | z_i\neq z_j \text{ for } i\neq j\}, n>1$, and with

$$\Omega_n=\sum_{1\le i<j\le n}\frac{t_{i,j}}{2\mathrm{i}\pi}\frac{d(z_i-z_j)}{z_i-z_j}=\sum_{1\le i<j\le n}\frac{t_{i,j}}{2\mathrm{i}\pi}d\log(z_i-z_j). \tag{12}$$

The equation (11) is completely integrable if one has

$$d\Omega_n-\Omega_n\wedge\Omega_n=0 \tag{13}$$

meaning that the noncommutative indeterminates $\{t_{i,j}\}_{i,j\ge1}$ must satisfy relations of infinitesimal braids [4]

$$\begin{cases} t_{i,j} &= 0 & \text{for} & i=j,\\ t_{i,j} &= t_{j,i} & \text{for} & i\neq j,\\ [t_{i,j},t_{i,k}+t_{j,k}] &= 0 & \text{for} & i\neq j\neq k,\\ [t_{i,j},t_{k,l}] &= 0 & \text{for} & i\neq j\neq k\neq l. \end{cases}$$

For the case $n=3$, in [10, 15, 16, 17], a theory have been developped in parallel with the group of group-like series[6] (in infinite dimension), for which local coordinates (of the second kind) are convergent polyzetas (or MZV and eMZV, see [12]). These studies allow to justify and to valide, on the one hand, the renormalisation of associators considered as noncommutative generating series of polyzetas, equipped with their double structure of shuffle and, on the other hand, the description of the graded kernel of the zeta function by a special method of identification of local coordinates developped in [15, 16] and implemented in Maple [2, 9].

The perspective being to connect with, on the one hand, the Lie algebra of Drinfel'd – Kohno [20, 21] which appears in the quantification of certain algebraic structures and which establishes an equivalence between two families of representations of the braid group of Artin [4], and on the other hand, the permutoassociahedron [19]. provided by MacLane's coherence theorem for associativities and commutativities in monoidal categories [4]. The work will be accompanied and supported by implementations usable by other users. In particular, Sage (or Maple) skills would be appreciated for implementation and experimentation on homogeneous algebras.

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