PhD proposal

Title: the combinatorics of some families of lattice polytopes
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Outline

Polytopes are convex hulls of finite subsets of $\mathbb{R}^d$. A polytope is called a *lattice polytope* when its vertices all belong to the lattice $\mathbb{Z}^d$. These objects appear naturally in mathematical optimization, in discrete geometry, in combinatorics, in theoretical physics, or in relation with toric varieties.

There are a number of deep, unanswered questions about the structure of the face lattice of polytopes, and especially, about its combinatorics. For instance, it is as yet not known whether, for some integer k, the number of the k-dimensional faces of a d-dimensional polytope can be less than both the number of its facets (its faces of dimension $d-1$) and the number of its vertices. This and other combinatorial questions are particularly relevant to the complexity of the simplex algorithm (see for instance [9]), that is widely used in practice, and gives rise to oustanding theoretical questions such as Smale's ninth problem [7].

This PhD proposal aims at gaining new understanding on the combinatorics of the face lattice of polytopes along (but not limited to) the following lines.

(1) When a d-dimensional polytope P is centrally-symmetric, it is conjectured by Kalai [5] that P has at least 3^d non-empty faces. This question has been solved when P is simple (i.e. when all of its vertices are contained in exactly d of its facets) by Stanley [8] and when $d = 4$ by Sanyal, Werner, and Ziegler [6]. Investivating the yet unsolved case when P is a two-level polytope [1] (a class of 0/1-polytopes that attracts a lot of attention from the combinatorial optimization community) is one of the main aims of the proposal.

(2) When P and Q are two non-homothetic d-dimensional polytopes, what can we say about the least number of faces of their Minkowski sum $P+Q$? Several flavors of this question can be investigated. For instance, in relation to the proof of Theorem 4.1 in [4], even a partial answer to this question in very specific cases may improve the known lower bound on the number of vertices of certain primitive zonotopes, a family a lattice polytopes with a number of extremal properties [2], or equivalently on the number of maximal unbalanced

families of subsets of $\{1, 2, ..., d+1\}$ [4]. Note that, when Q is a symmetric of P with respect to a point of the ambient space, this question is highly related to the conjecture stated in (1).

(3) The diameter of a polytope is defined as the diameter of the graph made up by its vertices and edges. Do there exist a sequence of polytopes $(P_n)_{n\geq 0}$ of diameter 1 and a sequence of polytopes $(Q_n)_{n\geq 0}$ of diameter 1 or 2 such that the diameter of the Minkowski sum $P_n + Q_n$ grows arbitrarily large when n goes to infinity? This question, posed in [3]

The candidate should have solid training in mathematics, in particular in combinatorics, geometry, combinatorial optimization, and in theoretical computer science. Knowledge of algebraic tools (algebraic topology/geometry) is a plus. Hard work and high dedication are expected during the PhD.

References

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