PhD proposal

Title: On the combinatorics of some families of lattice polytopes
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Outline

Polytopes are convex hulls of finite subsets of $\mathbb{R}^d$. A polytope is called a *lattice polytope* when its vertices all belong to the lattice $\mathbb{Z}^d$. These objects appear naturally in mathematical optimization, in discrete geometry, in combinatorics, in theoretical physics, or in relation with toric varieties.

There are a number of deep, unanswered questions about the structure of the face lattice of polytopes, and especially, about its combinatorics. For instance, it is as yet not known whether, for some integer k, the number of the k-dimensional faces of a d-dimensional polytope can be less than both the number of its facets (its faces of dimension $d-1$) and the number of its vertices. This and other combinatorial questions are particularly relevant to the complexity of a number of algorithms, such as the simplex algorithm, that is widely used in computer science and in all varieties of engineering (see Smale's ninth problem [17]).

This PhD proposal aims at gaining new understanding on the combinatorics of the face lattice of polytopes, using the computer as a tool. Here is a (non-exhaustive) list of the problems that may be investigated.

(1) When a d-dimensional polytope P is centrally-symmetric, it is conjectured by Kalai [12] that P has at least 3^d non-empty faces. This question has been solved when P is simple (i.e. when all of its vertices are contained in exactly d of its facets) [18] and when $d = 4$ [16]. On this topic, two questions seem within reach, especially with the use of the computer and the tools developed in [5, 14]. The first question is whether Kalai's conjecture holds for two-level polytopes, a class of 0/1-polytopes that attracts a lot of attention from the combinatorial optimization community [4]. The second question is the existence of d-dimensional polytopes with few (close to 3^d) non-empty faces that are neither Hanner [10], nor Hansen [11] polytopes.

(2) When P and Q are two non-homothetic d-dimensional polytopes, what can we say about the least number of faces of their Minkowski sum $P+Q$? Several flavors of this question can be investigated. For instance, in relation to the

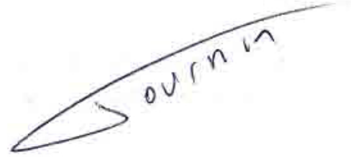
proof of Theorem 4.1 in [8], even a partial answer to this question in very specific cases may improve the known lower bound on the number of vertices of certain primitive zonotopes, a family a lattice polytopes with a number of extremal properties [6], or equivalently on the number of maximal unbalanced families of subsets of $\{1, 2, ..., d+1\}$ [8]. Note that, when Q is a symmetric of P with respect to a point of the ambient space, this question is highly related to Kalai's conjecture stated in (1).

(3) The diameter of a polytope is defined as the diameter of the graph made up by its vertices and edges. The Hirsch conjecture, closely related to the complexity of the simplex algorithm, states that a d-dimensional polytope with n facets has diameter at most $n - d$. While this conjecture has been disproved [15], it is known to hold for lattice polytopes contained in the hypercube $[0, 1]^d$ [13]. A question this proposal aims to tackle is the Hirsch conjecture for the lattice polytopes contained in the hypercube $[0, 2]^d$.

(4) The asymptotic behavior of the lattice polygons contained in the square $[0, k]^2$ is well studied. For instance, the largest possible number of vertices they can have grows like $12 \cdot (k/2\pi)^{2/3}$ when $k \to \infty$ [1]. This result has been recently generalized to any dimension, in terms of diameter, for the special case of lattice zonotopes [9]. On the other hand, the random generation and average asymptotic behavior of the lattice polygons contained in $[0, k]^2$ has also been studied in [2, 3]. Another aim of this proposal is to design random generation tools for the lattice zonotopes contained in the hypercube $[0, k]^d$ and to provide asymptotic estimates for their average behavior. This will contribute to better understanding the combinatorics and gain insight on the average and extremal properties of lattice polytopes.

In order to tackle these questions, the successful candidate will use as a starting point the results obtained by Rado Rakotonarivo during his PhD [14], and the probabilistic and geometric tools he developed. The candidate should have solid training both in mathematics and computer science, in particular in discrete geometry, probabilities, combinatorics, and combinatorial optimization.

Lionel Pournin



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